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TABLES FOR THE EXTREME ROOTS OF THE WISHART MATRIX

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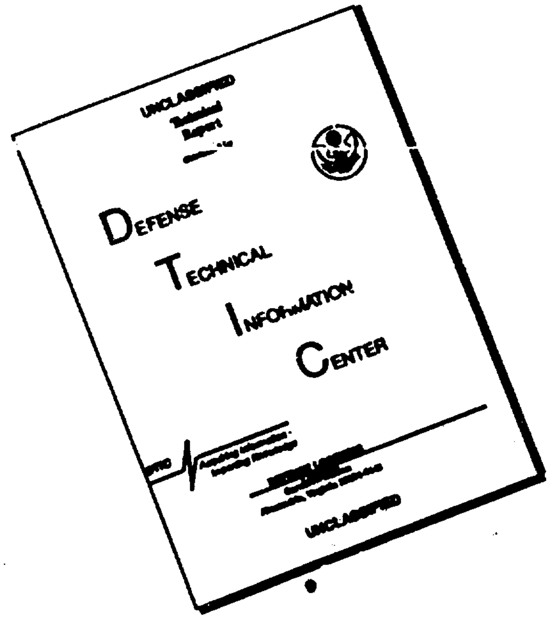

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13. ABSTRACT

Let ℓ_1 and ℓ_p be respectively the smallest and largest roots of the central pxp Wishart matrix with n degrees of freedom. In this report, the authors gave tables for the exact values of U for p = 2(1)10(2)20, α = 0.05, 0.025, 0.01, 0.005 and different values of n where

$$P[U^{-1} < \ell_1 < \ell_p < U] = (1-2\alpha).$$

Also, exact lower 1%, 2.5%, 5%, 10% points of the distribution of ℓ_p are given for p = 2(1)10(2)20 and for different values of n.

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FOREWORD

This report was prepared for Applied Mathematics Research Laboratory, Aerospace Research Laboratories by D. S. Clemm, P. R. Krishnaiah and V. B. Waikar under Project 7071, "Research in Applied Mathematics". The work of V. B. Waikar was performed at the Aerospace Research Laboratories while in the capacity of an Ohio State University Research Foundation Visiting Research Associate under Contract F 33615-67-C-1758. The present affiliation of V. B. Waikar is Miami University, Oxford, Ohio.

In this report, the authors gave tables for the exact percentage points of the joint distribution of the extreme roots as well as the tables for the exact lower percentage points of the largest root of the Wishart matrix.

The authors wish to thank Miss Eva Brandenburg for typing the manuscript.

ABSTRACT

Let ℓ_1 and ℓ_p be respectively the smallest and largest roots of the central pxp Wishart matrix with n degrees of freedom. In this report, the authors gave tables for the exact values of U for $p = 2(1)10(2)20$, $\alpha = 0.05$, 0.025, 0.01, 0.005 and different values of n where

$$P[U^{-1} \leq \ell_1 \leq \ell_p \leq U] = (1-2\alpha).$$

Also, exact lower 1%, 2.5%, 5%, 10% points of the distribution of ℓ_p are given for $p = 2(1)10(2)20$ and for different values of n.

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I. INTRODUCTION

Krishnaiah and Chang [3] gave an expression for the probability integral associated with the joint density of the extreme roots of the Wishart matrix. For computational purposes, this expression is significantly better than other expressions given in the literature ([1], [6]). Using this expression, the authors constructed tables for the percentage points associated with the joint density of the extreme roots of the Wishart matrix by imposing the condition that the lower critical value is equal to the reciprocal of the upper critical value. These tables are useful in the application of Roy's test for testing the hypothesis that the covariance matrix of a multivariate normal population is equal to a specified value. Some computations are also made to check for the accuracy of the approximate tables constructed by Hanumara and Thompson [2] by putting a different restriction on the choice of the critical values. Also tables of the lower percentage points of the largest root of the Wishart matrix are constructed in this paper.

2. PROBABILITY INTEGRAL OF THE JOINT DENSITY OF THE EXTREME ROOTS

Let X be a pxn matrix whose columns are independently distributed as multivariate normal with zero mean vector and covariance matrix I_p, the identity matrix. Also let $S = XX'$ and let $\ell_1 < \ell_2 < \ldots < \ell_p$ be the latent roots of S. Then it is well known that S has Wishart distribution and the joint density of $\ell_1, \ldots, \ell_p$ is given by

$$f_1(\ell_1,\ldots,\ell_p) = k(p,n) \prod_{i=1}^{p} [\ell_i^r \exp(-\ell_i/2) \prod_{i>j}^{p} (\ell_i - \ell_j) \tag{2.1}$$

$$0<\ell_1<\ldots<\ell_p<\infty$$

where $r = (n-p-1)/2$ and

$$k(p,n) = \pi^{p/2} (1/2)^{np/2} \prod_{i=1}^{p} [\Gamma(n+1-i)/2)\Gamma((p+1-i)/2].$$

Krishnaiah and Chang [3] gave an exact expression for the probability integral of the joint density of ℓ_1 and ℓ_p which is given by

$$P[L \leq \ell_1 < \ell_p \leq U] = k(p,n)\rho(\Psi;\ p,r,L,U) \tag{2.2}$$

where

$$\rho(\psi;\ p,r,L,U) = \begin{cases} \Delta(\Psi;\ 2m,r,L,U) \text{ when } p = 2m & (2.3) \\ \sum_{i=0}^{2m} (-1)^i\ F_{r+i}(U) \quad G_{i+1}(\Psi;\ 2m+1,r,L,U) & \\ \qquad\qquad \text{when } p = 2m+1 & (2.4) \end{cases}$$

and

$$\Delta(\Psi;\ 2m,r,L,U) = |(a_{ij})_{i,\ j=1,\ldots,2m}|^{1/2},$$

$$G_t(\Psi;\ 2m+1,r,L,U) = |(a_{ij})_{i,j=1,\ldots,\ t-1,\ t+1,\ldots,2m+1}|^{1/2} \text{ for } p>1$$

$$a_{ij} = f_{i+r-1}^{j+r-1}, \quad f_s^t = F_s^t - F_t^s,$$

$$F_s^t = \int_L^U F_s(\theta)\ \theta^t\ \Psi(\theta)d\theta, \quad F_s(\theta) = \int_L^\theta \Psi(x)\ x^s\ dx,$$

$\Psi(x) = \exp(-x/2)$. Also, let $G_1(\psi;\ 1,r,L,U)\equiv 1$.

In this paper, the authors give tables for the exact values of U for $\alpha = 0.05,\ 0.025, 0.01,\ 0.005$, $p = 2(1)10(2)20$, $n = (p+1)(1)20(2)30(5)50$ where

$$P[U^{-1} \leq \ell_1 \leq \ell_p \leq U] = (1-2\alpha). \tag{2.5}$$

The authors also gave lower 1%, 2.5%, 5% and 10% values of the distribution of ℓ_p for $p = 2(1)10(2)20$, $n = (p+1)\ (1)20(2)30(5)50$ by using the following known formula [3]:

$$P[\ell_p > U] = 1-k(p,n)\rho(\psi;\ p,r,0,U)$$

Hanumara and Thompson [2] constructed tables for L and U for $p = 2(1)$ 10 and various values of r and α satisfying

$$P[\ L \leq \ell_1 < \ell_p \leq U] = 1-2\alpha \tag{2.6}$$

and

$$P[\ \ell_1 \geq L] = 1 - \alpha. \tag{2.7}$$

For $p = 2$, Thompson [7] computed exact values of L and U. For $p \geq 3$, Hanumara and Thompson [2] state that exact values are very difficult to compute and so they computed approximate values L_1 and U_1 of L and U respectively where L_1 and U_1 satisfy

$$\begin{aligned} &P[\ \ell_1 \geq L_1] = (1 - \alpha) \\ &P[\ \ell_p \leq U_1] = (1 - \alpha). \end{aligned} \tag{2.8}$$

For $p = 2$ and 3, they found empirically that L_1 and U_1 are close to L and U respectively. For higher values of $p(\leq 10)$ we computed L and U exactly for some typical values of n and α and found them to be close to L_1 and U_1 given in [2].

3. APPLICATIONS

Let H_0: $\Sigma = \Sigma_0$, H_a: $\Sigma \neq \Sigma_0$ where Σ is the covariance matrix of a p-variate normal distribution and Σ_0 is known. Further let S/n be the sample covariance matrix based on a sample of size n+1. According to the procedure of

Roy [5, p. 30] an acceptance region for testing H_0 against H_a is given by

$$L \leq \ell_1 < \ell_p \leq U \tag{3.1}$$

where $\ell_1 < \ell_2 < \ldots < \ell_p$ are the characteristic roots of $S\Sigma_0^{-1}$ and L<U are constants such that

$$P[\, L \leq \ell_1 < \ell_p \leq U \mid H_0] = 1 - 2\alpha. \tag{3.2}$$

Note that under H_0, $S\Sigma_0^{-1}$ has a Wishart distribution $W(I,n)$. Now, the optimum choice of L and U (in the sense of maximizing the power) is not known. Thus one simple way of choosing L and U is to put L = 1/U and then use the tables given in this paper.

One can also construct simultaneous confidence intervals for the elements σ_{ij} of the covariance matrix Σ using the tables given here. Roy [5, p. 106] showed that

$$U^{-1}\, \underset{\sim}{a}'\, S\, \underset{\sim}{a} \leq \underset{\sim}{a}'\, \Sigma\, \underset{\sim}{a} \leq L^{-1}\, \underset{\sim}{a}'\, S\, \underset{\sim}{a} \tag{3.3}$$

is a set of simultaneous confidence bounds on $\underset{\sim}{a}'\, \Sigma\, \underset{\sim}{a}$ for all arbitrary nonnull vectors a with confidence coefficients $1 - 2\alpha$ provided L and U satisfy equation (2.6). By choosing $\underset{\sim}{a}$ appropriately in (3.3) one can obtain simultaneous confidence intervals for σ_{ij}. Thompson [7] derived the following simultaneous confidence intervals for σ_{ij} using Roy's results:

$$\begin{aligned} &U^{-1}\, s_{ii} \leq \sigma_{ii} \leq L^{-1}\, s_{ii} \quad i = 1,\ldots,p \\ &\left|\sigma_{ij} - \tfrac{1}{2}(U^{-1} + L^{-1}) s_{ij}\right| \leq \tfrac{1}{2}(L^{-1} - U^{-1})\, (s_{ii}\, s_{jj})^{1/2} \quad i \neq j. \end{aligned} \tag{3.4}$$

Thus (3.4) represents simultaneous confidence intervals for σ_{ij}, $i,j=1,\ldots,p$

with confidence coefficient $1 - 2\alpha$ where L and U satisfy (2.6) and $S = (s_{ij})$. Hence putting $L = 1/U$ and using Table I, one can construct exact simultaneous confidence intervals for σ_{ij} given by (3.4).

As a further application, we mention that these tables are useful in obtaining the simultaneous confidence intervals for the variance components of the two-way layout with unequal variances. Since this application is discussed in detail by Thompson [7,8] and Hanumara and Thompson [2], we omit its discussion here.

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TABLE I*
PERCENTAGE POINTS OF THE JOINT DISTRIBUTION
OF THE EXTREME ROOTS

P= 2

n \ α	.05	.025	.01	.005
3	12.49	19.98	49.50	99.50
4	11.27	13.39	16.50	19.63
5	12.58	14.57	17.10	18.99
6	14.15	16.22	18.79	20.65
7	15.72	17.87	20.55	22.45
8	17.26	19.49	22.25	24.23
9	18.75	21.07	23.91	25.95
10	20.21	22.60	25.54	27.63
11	21.66	24.12	27.13	29.27
12	23.09	25.61	28.70	30.89
13	24.49	27.08	30.24	32.47
14	25.88	28.54	31.76	34.05
15	27.25	29.96	33.25	35.59
16	28.61	31.39	34.74	37.12
17	29.96	32.80	36.20	38.63
18	31.31	34.19	37.67	40.13
19	32.64	35.57	39.10	41.61
20	33.96	36.94	40.53	43.08
22	36.57	39.66	43.36	45.98
24	39.17	42.35	46.16	48.85
26	41.74	45.00	48.92	51.69
28	44.27	47.64	51.65	54.48
30	46.80	50.24	54.35	57.25
35	53.04	56.69	61.02	64.05
40	59.19	63.02	67.57	70.75
45	65.27	69.27	74.02	77.34
50	71.29	75.46	80.40	83.84

TABLE 1 (Continued)

P= 3

n \ α	.05	.025	.01	.005
4	17.70	29.34	74.25	149.25
5	15.66	18.08	21.69	25.70
6	16.99	19.19	21.94	23.99
7	18.64	20.89	23.67	25.66
8	20.31	22.62	25.48	27.52
9	21.92	24.32	27.25	29.34
10	23.59	25.97	28.97	31.12
11	25.06	27.58	30.67	32.86
12	26.59	29.18	32.33	34.56
13	28.09	30.73	33.96	36.25
14	29.57	32.29	35.57	37.89
15	31.04	33.80	37.14	39.51
16	32.50	35.31	38.72	41.13
17	33.93	36.80	40.25	42.71
18	35.35	38.28	41.80	44.28
19	36.77	39.74	43.30	45.83
20	38.17	41.18	44.82	47.38
22	40.94	44.04	47.78	50.42
24	43.68	46.87	50.70	53.41
26	46.39	49.67	53.59	56.35
28	49.06	52.43	56.44	59.28
30	51.72	55.15	59.26	62.15
35	58.26	61.89	66.22	69.24
40	64.71	68.50	73.02	76.18
45	71.05	75.01	79.71	83.03
50	77.33	81.44	86.33	89.73

*The entries in the table give the values of U where

$$k(p,n) \int \cdots \int_{U^{-1} \le \ell_1 \le \cdots \le \ell_p \le U} \prod_{i=1}^{p} [\ell_i^r \exp(-\ell_i/2)] \prod_{i>j} (\ell_i - \ell_j) d\ell_1 \cdots d\ell_p = 1-2\alpha.$$

TABLE I (Continued)

P= 4

n	α .05	.025	.01	.005
5	22.75	39.01	99.00	199.00
6	19.94	22.61	26.65	31.63
7	21.27	23.63	26.60	28.79
8	22.98	25.39	28.33	30.43
9	24.70	27.16	30.17	32.32
10	26.36	28.90	31.98	34.18
11	28.01	30.60	33.76	36.00
12	29.63	32.28	35.50	37.77
13	31.21	33.92	37.19	39.52
14	32.77	35.54	38.88	41.25
15	34.32	37.13	40.53	42.94
16	35.85	38.71	42.16	44.60
17	37.35	40.27	43.78	46.25
18	38.84	41.81	45.36	47.89
19	40.33	43.33	46.95	49.51
20	41.79	44.84	48.51	51.10
22	44.68	47.83	51.59	54.26
24	47.55	50.77	54.63	57.36
26	50.36	53.67	57.62	60.41
28	53.16	56.54	60.57	63.42
30	55.92	59.36	63.49	66.40
35	62.72	66.34	70.67	73.71
40	69.39	73.18	77.69	80.86
45	75.95	79.90	84.60	87.88
50	82.43	86.53	91.40	94.79

TABLE I (Continued)

P= 5

n	α .05	.025	.01	.005
6	27.70	48.74	123.75	248.75
7	24.17	27.05	31.49	37.49
8	25.49	28.01	31.14	33.45
9	27.23	29.76	32.87	35.08
10	28.98	31.58	34.74	36.98
11	30.70	33.36	36.58	38.86
12	32.39	35.10	38.38	40.70
13	34.05	36.80	40.15	42.52
14	35.67	38.50	41.90	44.31
15	37.29	40.15	43.61	46.06
16	38.89	41.79	45.29	47.78
17	40.44	43.41	46.97	49.49
18	42.00	45.00	48.62	51.18
19	43.54	46.59	50.24	52.83
20	45.06	48.15	51.87	54.42
22	48.07	51.24	55.05	57.73
24	51.02	54.28	58.18	60.93
26	53.95	57.27	61.25	64.07
28	56.83	60.23	64.30	67.18
30	59.68	63.16	67.31	70.24
35	66.70	70.34	74.69	77.75
40	73.57	77.37	81.91	85.07
45	80.37	84.28	88.98	92.25
50	86.99	91.09	95.95	99.34

TABLE I (Continued)

P= 6

n \ α	.05	.025	.01	.005
7	32.58	58.49	148.49	298.50
8	28.36	31.43	36.24	43.37
9	29.67	32.32	35.61	38.04
10	31.44	34.09	37.33	39.62
11	33.21	35.92	39.21	41.54
12	34.97	37.74	41.07	43.44
13	36.69	39.51	42.91	45.32
14	38.38	41.25	44.71	47.15
15	40.05	42.96	46.47	48.96
16	41.70	44.66	48.21	50.73
17	43.31	46.33	49.94	52.50
18	44.92	47.97	51.63	54.23
19	46.52	49.61	53.31	55.94
20	48.08	51.22	54.98	57.64
22	51.19	54.41	58.25	60.97
24	54.24	57.54	61.47	64.25
26	57.25	60.62	64.64	67.48
28	60.22	63.66	67.76	70.66
30	63.16	66.67	70.85	73.79
35	70.37	74.03	78.41	81.48
40	77.43	81.24	85.79	88.98
45	84.35	88.31	93.03	96.33
50	91.17	95.27	100.15	103.55

TABLE I (Continued)

P= 7

n \ α	.05	.025	.01	.005
8	37.40	68.24	173.24	348.25
9	32.54	35.77	40.92	49.29
10	33.82	36.60	40.02	42.56
11	35.61	38.38	41.74	44.12
12	37.40	40.23	43.64	46.04
13	39.19	42.05	45.52	47.96
14	40.94	43.85	47.36	49.84
15	42.65	45.62	49.18	51.70
16	44.36	47.36	50.98	53.53
17	46.03	49.08	52.75	55.33
18	47.68	50.78	54.48	57.11
19	49.32	52.45	56.21	58.86
20	50.94	54.11	57.92	60.60
22	54.13	57.38	61.27	64.02
24	57.26	60.60	64.57	67.38
26	60.36	63.76	67.82	70.68
28	63.40	66.[illegible]	71.02	73.92
30	66.42	69.95	74.17	77.13
35	73.81	77.49	81.90	84.98
40	81.02	84.87	89.44	92.64
45	88.11	92.09	96.82	100.13
50	95.08	99.19	104.08	107.49

TABLE I (Continued)

P= 8

n \ α	.05	.025	.01	.005
9	42.19	77.98	197.99	398.00
10	36.69	40.06	45.54	55.28
11	37.96	40.84	44.39	47.04
12	39.75	42.63	46.09	48.56
13	41.57	44.49	48.01	50.49
14	43.38	46.34	49.91	52.43
15	45.14	48.15	51.77	54.33
16	46.89	49.94	53.60	56.20
17	48.61	51.71	55.41	58.04
18	50.30	53.45	57.20	59.85
19	51.99	55.16	58.97	61.66
20	53.65	56.87	60.71	63.43
22	56.92	60.21	64.15	66.92
24	60.14	63.51	67.52	70.36
26	63.31	66.74	70.84	73.73
28	66.42	69.93	74.11	77.04
30	69.51	73.07	77.32	80.31
35	77.06	80.78	85.21	88.32
40	84.43	88.29	92.89	96.11
45	91.66	95.66	100.42	103.74
50	98.76	102.89	107.80	111.22

TABLE I (Continued)

P= 9

n \ α	.05	.025	.01	.005
10	46.95	87.73	222.74	447.75
11	40.83	44.34	50.12	51.35
12	42.08	45.06	48.74	51.48
13	43.88	46.85	50.43	52.97
14	45.72	48.73	52.34	54.90
15	47.54	50.59	54.25	56.84
16	49.32	52.42	56.13	58.75
17	51.09	54.23	57.99	60.64
18	52.83	56.01	59.82	62.50
19	54.55	57.77	61.61	64.34
20	56.26	59.51	63.40	66.15
22	59.60	62.93	66.91	69.72
24	62.90	66.29	70.35	73.21
26	66.13	69.60	73.73	76.65
28	69.32	72.85	77.07	80.02
30	72.47	76.06	80.34	83.36
35	80.17	83.92	88.38	91.51
40	87.69	91.57	96.20	99.44
45	95.05	99.07	103.84	107.18
50	102.28	106.42	111.35	114.79

TABLE I (Continued)

P=10

n \ α	.05	.025	.01	.005
11	51.68	97.48	247.49	497.50
12	44.96	48.50	54.66	67.50
13	46.19	49.26	53.05	55.87
14	48.00	51.05	54.74	57.35
15	49.85	52.94	56.66	59.29
16	51.68	54.82	58.57	61.24
17	53.48	56.66	60.47	63.16
18	55.27	58.49	62.33	65.05
19	57.03	60.28	64.17	66.92
20	58.76	62.05	66.00	68.76
22	62.19	65.56	69.57	72.41
24	65.55	68.98	73.08	75.96
26	68.85	72.34	76.52	79.46
28	72.10	75.67	79.91	82.90
30	75.31	78.94	83.26	86.29
35	83.16	86.93	91.43	94.58
40	90.81	94.72	99.38	102.63
45	98.30	102.34	107.15	110.50
50	105.66	109.82	114.77	118.21

TABLE I (Continued)

P=12

n \ α	.05	.025	.01	.005
13	61.07	116.97	296.99	596.99
14	53.19	57.07	63.66	79.99
15	54.38	57.63	61.63	64.62
16	56.20	59.43	63.29	66.03
17	58.07	61.33	65.22	67.97
18	59.93	63.21	67.15	69.93
19	61.76	65.08	69.07	71.87
20	63.57	66.93	70.95	73.79
22	67.13	70.56	74.65	77.55
24	70.61	74.11	78.28	81.22
26	74.04	77.60	81.85	84.84
28	77.41	81.03	85.35	88.39
30	80.73	84.41	88.80	91.88
35	88.85	92.67	97.23	100.42
40	96.75	100.71	105.42	108.71
45	104.48	108.56	113.42	116.81
50	112.05	116.25	121.25	124.73

TABLE I (Continued)

P=14

n \ α	.05	.025	.01	.005
15	70.41	136.47	346.49	696.49
16	61.40	65.49	72.59	92.65
17	62.55	65.95	70.15	73.27
18	64.37	67.75	71.79	74.65
19	66.26	69.67	73.72	76.59
20	68.13	71.57	75.67	78.56
22	71.82	75.31	79.49	82.43
24	75.42	78.98	83.24	86.22
26	78.96	82.58	86.91	89.94
28	82.44	86.12	90.51	93.60
30	85.87	89.60	94.07	97.19
35	94.23	98.11	102.72	105.96
40	102.36	106.37	111.14	114.47
45	110.31	114.43	119.34	122.76
50	118.08	122.32	127.37	130.88

TABLE I (Continued)

P=16

n \ α	.05	.025	.01	.005
17	79.71	155.97	395.99	795.99
18	69.58	73.87	81.42	105.38
19	70.70	74.24	78.61	81.87
20	72.53	76.04	80.23	83.20
22	76.31	79.88	84.13	87.12
24	80.03	83.66	87.98	91.01
26	83.67	87.35	91.75	94.83
28	87.25	90.99	95.46	98.58
30	90.77	94.57	99.10	102.26
35	99.37	103.30	107.98	111.24
40	107.72	111.77	116.60	119.96
45	115.86	120.03	124.99	128.44
50	123.83	128.11	133.20	136.75

TABLE I (Continued)

P=18

n \ α	.05	.025	.01	.005
19	88.98	175.46	445.48	895.49
20	77.75	82.22	90.20	118.15
22	80.66	84.30	88.64	91.71
24	84.47	88.16	92.55	95.63
26	88.21	91.96	96.42	99.54
28	91.89	95.68	100.21	103.38
30	95.51	99.35	103.94	107.15
35	104.32	108.29	113.03	116.35
40	112.86	116.97	121.84	125.24
45	121.19	125.40	130.42	133.91
50	129.34	133.66	138.80	142.37

TABLE I (Continued)

P=20

n \ α	.05	.025	.01	.005
22	85.90	90.55	98.94	130.93
24	88.79	92.54	97.03	100.19
26	92.61	96.41	100.94	104.10
28	95.38	100.23	104.82	108.03
30	100.08	103.99	108.54	111.99
35	109.09	113.13	117.92	121.26
40	117.83	121.98	126.91	130.35
45	126.34	130.60	135.66	139.18
50	134.65	139.01	144.19	147.80

TABLE II**
LOWER PERCENTAGE POINTS OF THE LARGEST ROOT

P= 2

n \ α	.01	.025	.05	.10
3	.69	.99	1.32	1.78
4	1.25	1.66	2.08	2.68
5	1.86	2.36	2.89	3.59
6	2.52	3.12	3.72	4.51
7	3.21	3.90	4.57	5.45
8	3.93	4.69	5.43	6.39
9	4.67	5.50	6.30	7.34
10	5.42	6.31	7.18	8.29
11	6.18	7.14	8.07	9.24
12	6.96	7.99	8.97	10.20
13	7.74	8.84	9.87	11.16
14	8.53	9.69	10.77	12.12
15	9.33	10.54	11.67	13.08
16	10.15	11.40	12.58	14.05
17	10.97	12.27	13.49	15.01
18	11.79	13.14	14.41	15.98
19	12.62	14.02	15.32	16.95
20	13.44	14.90	16.24	17.92
22	15.13	16.67	18.09	19.85
24	16.82	18.45	19.95	21.80
26	18.53	20.23	21.81	23.74
28	20.24	22.03	23.68	25.69
30	21.97	23.84	25.55	27.64
35	26.32	28.37	30.24	32.52
40	30.73	32.95	34.96	37.41
45	35.16	37.55	39.70	42.30
50	39.64	42.16	44.45	47.20

TABLE II (Continued)

P= 3

n \ α	.01	.025	.05	.10
4	2.31	2.87	3.42	4.15
5	3.18	3.84	4.48	5.32
6	4.07	4.83	5.54	6.47
7	4.98	5.81	6.60	7.61
8	5.90	6.80	7.65	8.73
9	6.82	7.79	8.70	9.85
10	7.75	8.79	9.75	10.96
11	8.69	9.78	10.80	12.07
12	9.64	10.78	11.84	13.17
13	10.58	11.77	12.88	14.27
14	11.52	12.77	13.92	15.36
15	12.47	13.76	14.96	16.46
16	13.41	14.76	16.00	17.54
17	14.36	15.76	17.04	18.63
18	15.31	16.76	18.08	19.71
19	16.27	17.75	19.11	20.79
20	17.23	18.75	20.15	21.87
22	19.14	20.75	22.21	24.02
24	21.06	22.74	24.28	26.17
26	22.98	24.74	26.34	28.30
28	24.91	26.74	28.40	30.44
30	26.84	28.74	30.46	32.57
35	31.67	33.73	35.60	37.87
40	36.52	38.73	40.73	43.15
45	41.37	43.73	45.85	48.42
50	46.23	48.72	50.96	53.67

*The entries in the table are the values of U where

$$P[\ell_p \geq U] = 1-k(p,n)\int_{0\leq\ell_1<\dots<\ell_p<U}\cdots\int \prod_{i=1}^{r}[\ell_i^r \exp(-\ell_i/2)]\prod_{i>j}(\ell_i-\ell_j)\,d\ell_1\cdots d\ell_p = 1-\alpha$$

TABLE II (Continued)

P= 4

n	α .01	.025	.05	.10
5	4.47	5.23	5.98	6.93
6	5.56	6.40	7.21	8.24
7	6.64	7.56	8.43	9.53
8	7.71	8.70	9.63	10.81
9	8.78	9.83	10.82	12.06
10	9.85	10.96	12.00	13.29
11	10.91	12.07	13.16	14.52
12	11.97	13.19	14.32	15.74
13	13.02	14.30	15.48	16.94
14	14.08	15.40	16.63	18.14
15	15.13	16.51	17.77	19.33
16	16.19	17.61	18.91	20.51
17	17.24	18.71	20.04	21.69
18	18.29	19.80	21.17	22.87
19	19.34	20.89	22.30	24.03
20	20.39	21.98	23.42	25.20
22	22.49	24.14	25.66	27.51
24	24.58	26.31	27.89	29.81
26	26.67	28.46	30.10	32.10
28	28.75	30.61	32.31	34.37
30	30.83	32.76	34.51	36.64
35	36.02	38.10	39.98	42.27
40	41.19	43.42	45.42	47.85
45	46.36	48.72	50.84	53.40
50	51.52	54.00	56.23	58.92

TABLE II (Continued)

P= 5

n	α .01	.025	.05	.10
6	6.98	7.91	8.80	9.91
7	8.19	9.19	10.14	11.33
8	9.39	10.47	11.47	12.73
9	10.58	11.72	12.78	14.09
10	11.76	12.96	14.06	15.43
11	12.93	14.18	15.33	16.75
12	14.10	15.39	16.59	18.07
13	15.26	16.60	17.84	19.36
14	16.41	17.80	19.07	20.65
15	17.56	18.99	20.30	21.92
16	18.70	20.17	21.52	23.18
17	19.83	21.35	22.74	24.44
18	20.97	22.52	23.95	25.69
19	22.10	23.69	25.14	25.93
20	23.23	24.86	26.34	28.15
22	25.46	27.16	28.72	30.61
24	27.69	29.47	31.08	33.04
26	29.91	31.76	33.42	35.46
28	32.12	34.03	35.75	37.85
30	34.33	36.29	38.07	40.23
35	39.81	41.92	43.83	46.13
40	45.26	47.50	49.52	51.97
45	50.69	53.05	55.18	5[illegible].75
50	56.09	58.57	60.80	63.49

TABLE II (Continued)

P= 6

n	α .01	.025	.05	.10
7	9.69	10.78	11.78	13.04
8	11.00	12.15	13.21	14.54
9	12.30	13.51	14.62	16.00
10	13.58	14.85	16.00	17.44
11	14.85	16.15	17.36	18.85
12	16.10	17.46	18.71	20.24
13	17.35	18.75	20.03	21.62
14	18.59	20.02	21.35	22.98
15	19.81	21.29	22.65	24.32
16	21.03	22.55	23.95	25.66
17	22.23	23.80	25.23	26.98
18	23.43	25.04	26.50	28.29
19	24.63	26.27	27.77	29.60
20	25.82	27.50	29.03	30.90
22	28.20	29.94	31.53	33.46
24	30.55	32.36	34.00	36.01
26	32.89	34.76	36.46	38.53
28	35.21	37.14	38.90	41.03
30	37.52	39.52	41.32	43.51
35	43.26	45.39	47.31	49.65
40	48.94	51.20	53.24	55.70
45	54.59	56.97	59.11	61.69
50	60.19	62.69	64.93	67.63

TABLE II (Continued)

P= 7

n	α .01	.025	.05	.10
8	12.57	13.78	14.89	16.27
9	13.96	15.22	16.38	17.83
10	15.32	16.65	17.85	19.34
11	16.67	18.04	19.29	20.84
12	18.01	19.42	20.72	22.30
13	19.34	20.79	22.11	23.75
14	20.64	22.13	23.50	25.18
15	21.93	23.47	24.87	26.59
16	23.21	24.79	26.23	27.99
17	24.49	26.10	27.57	29.37
18	25.76	27.40	28.91	30.75
19	27.01	28.70	30.23	32.11
20	28.25	29.99	31.55	33.46
22	30.75	32.54	34.16	36.13
24	33.21	35.06	36.74	38.78
26	35.65	37.56	39.29	41.40
28	38.08	40.05	41.83	43.99
30	40.48	42.51	44.34	46.55
35	46.45	48.61	50.56	52.91
40	52.34	54.63	56.58	59.16
45	58.19	60.58	62.74	65.34
50	63.98	66.48	68.74	71.45

TABLE II (Continued)

P= 8

n \ α	.01	.025	.05	.10
9	15.56	16.88	18.09	19.59
10	17.01	18.38	19.64	21.18
11	18.44	19.86	21.15	22.75
12	19.85	21.30	22.64	24.28
13	21.23	22.74	24.11	25.79
14	22.60	24.15	25.56	27.28
15	23.96	25.55	26.99	28.76
16	25.31	26.93	28.40	30.21
17	26.64	28.30	29.81	31.65
18	27.97	29.66	31.20	33.08
19	29.28	31.01	32.58	34.49
20	30.58	32.34	33.95	35.89
22	33.17	35.00	36.66	38.67
24	35.73	37.62	39.33	41.41
26	38.27	40.21	41.98	44.11
28	40.79	42.79	44.60	46.79
30	43.28	45.34	47.19	49.45
35	49.45	51.64	53.61	55.99
40	55.54	57.84	59.92	62.42
45	61.56	63.97	66.14	68.77
50	67.52	70.04	72.30	75.03

TABLE II (Continued)

P= 9

n \ α	.01	.025	.05	.10
10	18.64	20.06	21.36	22.95
11	20.15	21.61	22.95	24.59
12	21.62	23.12	24.50	26.19
13	23.07	24.62	26.03	27.75
14	24.51	26.09	27.54	29.31
15	25.92	27.55	29.03	30.84
16	27.32	28.99	30.50	32.34
17	28.71	30.41	31.96	33.84
18	30.09	31.82	33.39	35.31
19	31.46	33.22	34.83	36.78
20	32.81	34.61	36.24	38.23
22	35.49	37.35	39.04	41.09
24	38.14	40.07	41.81	43.92
26	40.77	42.75	44.54	46.71
28	43.36	45.40	47.24	49.46
30	45.94	48.03	49.92	52.20
35	52.31	54.52	56.51	58.92
40	58.57	60.90	62.99	65.52
45	64.75	67.19	69.38	72.02
50	70.87	73.41	75.69	78.43

TABLE II (Continued)

P=10

n \ α	.01	.025	.05	.10
11	21.81	23.31	24.70	26.38
12	23.34	24.90	26.31	28.04
13	24.86	26.45	27.90	29.67
14	26.34	27.98	29.46	31.27
15	27.82	29.49	31.00	32.85
16	29.27	30.98	32.52	34.41
17	30.72	32.45	34.03	35.95
18	32.14	33.91	35.52	37.47
19	33.55	35.36	36.99	38.98
20	34.96	36.79	38.45	40.48
22	37.73	39.63	41.34	43.43
24	40.46	42.42	44.19	46.33
26	43.17	45.18	47.00	49.20
28	45.85	47.91	49.78	52.03
30	48.50	50.62	52.52	54.84
35	55.03	57.27	59.29	61.73
40	61.46	63.82	65.93	68.48
45	67.80	70.26	72.46	75.12
50	74.06	76.62	78.92	81.68

TABLE II (Continued)

P=12

n \ α	.01	.025	.05	.10
13	28.30	29.97	31.49	33.35
14	29.89	31.61	33.16	35.05
15	31.47	33.21	34.80	36.72
16	33.03	34.80	36.41	38.37
17	34.56	36.37	38.01	40.00
18	36.07	37.92	39.59	41.61
19	37.57	39.45	41.14	43.20
20	39.06	40.97	42.69	44.78
22	42.00	43.96	45.74	47.89
24	44.89	46.91	48.73	50.94
26	47.75	49.82	51.69	53.95
28	50.56	52.69	54.60	56.92
30	53.35	55.53	57.48	59.85
35	60.21	62.50	64.56	67.05
40	66.94	69.34	71.49	74.09
45	73.56	76.06	78.30	81.00
50	80.08	82.68	85.01	87.81

TABLE II (Continued)

P=14

n \ α	.01	.025	.05	.10
15	34.97	36.79	38.43	40.43
16	36.62	38.46	40.13	42.16
17	38.24	40.11	41.81	43.87
18	39.83	41.74	43.47	45.56
19	41.42	43.35	45.10	47.23
20	42.98	44.95	46.72	48.88
22	46.06	48.08	49.91	52.12
24	49.09	51.17	53.05	55.31
26	52.08	54.21	56.13	58.45
28	55.03	57.21	59.17	61.54
30	57.95	60.17	62.17	64.59
35	65.10	67.44	69.54	72.07
40	72.09	74.54	76.73	79.37
45	78.97	81.51	83.79	86.53
50	85.73	88.37	90.73	93.56

TABLE II (Continued)

P=16

n \ α	.01	.025	.05	.10
17	41.78	43.72	45.47	47.60
18	43.45	45.42	47.20	49.36
19	45.10	47.10	48.91	51.10
20	46.75	48.77	50.60	52.81
22	49.97	52.04	53.92	56.19
24	53.13	55.26	57.18	59.50
26	56.24	58.42	60.39	62.75
28	59.31	61.54	63.54	65.97
30	62.33	64.61	66.66	69.13
35	69.76	72.14	74.28	76.85
40	77.00	79.49	81.72	84.40
45	84.10	86.69	89.00	91.78
50	91.09	93.76	96.15	99.03

TABLE II (Continued)

P=18

n	α .01	.025	.05	.10
19	48.69	50.75	52.60	54.84
20	50.39	52.47	54.35	56.62
22	53.74	55.88	57.80	60.12
24	57.02	59.21	61.17	63.55
26	60.25	62.48	64.49	66.92
28	63.43	65.71	67.76	70.23
30	66.56	68.89	70.97	73.49
35	74.23	76.66	78.84	81.47
40	81.71	84.24	86.50	89.23
45	89.03	91.65	94.00	96.82
50	96.22	98.93	101.35	104.26

TABLE II (Continued)

P=20

n	α .01	.025	.05	.10
22	57.41	59.60	61.56	63.94
24	60.80	63.04	65.05	67.48
26	64.14	66.42	68.47	70.95
28	67.42	69.75	71.84	74.36
30	70.66	73.03	75.15	77.72
35	78.56	81.03	83.25	85.92
40	86.25	88.83	91.12	93.89
45	93.78	96.44	98.82	101.68
50	101.16	103.90	106.36	109.31